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Geometric Classification of General Dynamical Systems

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A combination of the geometric spectral theory (based on a pair of an order-unit space and a base-norm space) with the theory of invariant cones in Lie algebras can replace the associative $*$-algebras as a complete description of dynamical systems. This geometric language is equally applicable to the classical and quantum cases. Reversing the relation between the automorphism groups of the two relevant structures—(lattice) order and Lie product—one may obtain a large class of new (quantum) dynamical systems.

1. INTRODUCTION

In the last few years the theory of invariant cones in finite-dimensional real Lie algebras has been successfully developed (Paneitz, 1981, 1983; Hilgert and Hofmann, 1988; Hilgert *et al.*, 1988). By definition, such cones are invariant under the action of the inner Lie automorphisms and appear as a natural object in the context of Lie algebras, where there is no purely algebraic concept of positivity. Similar (in general, infinite-dimensional) invariant geometric structures can be found in the usual description of both classical and quantum dynamical systems (mechanical, statistical, field systems, etc.). They are exemplified by the cone of all nonnegative functions on the classical phase space (with the Poisson brackets as Lie product) and the cone of all positive self-adjoint operators (with the operator commutator), respectively.

These natural cones and the corresponding order relations are, of course, very well known, but they (and in particular their invariance properties) have never been used for the purpose of defining and classifying different kinds of dynamical systems. With the appearance of the geometric spectral theory (Alfsen and Shultz, 1976; Abbati and Manià, 1981; Riedel,

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1983), one should expect this situation to change: the geometry of the positive cones (and their duals) turned out to be the relevant structure which completely determines the spectral (i.e., statistical) properties of the dynamical variables.

2. INVARIANT GEOMETRIC DYNAMICAL SYSTEMS

I first exploit the potentialities of the abstract geometric spectral theory [order-unit and base-norm spaces in spectral duality in the sense of Alfsen and Shultz (1976)] in order to replace the usual (associative) algebraic language by a more economical (and physically complete) description. What one has to do is to combine the geometric spectral theory with an appropriate Lie algebra structure. In all known cases (classical and quantum) "appropriate" means invariance of the positive cone under the inner Lie automorphisms. Thus, one comes to the notion of "invariantly ordered spectral Lie algebras" as a geometric description (or definition) of a generic dynamical system. The term "spectral" refers to the properties required by the geometric spectral theory. The starting point is the following definition.

Definition (D). Let (A, A^+, e) and (V, V^+, K) be an order-unit and a base-norm space in spectral duality relative to the bilinear form $\langle a, \rho\rangle$, $a \in A$, $\rho \in V$. Here $e \in A^+$ is the order unit in A and $K \subset V^+$ is the base of the positive cone V^+ in V. Let $\{e_\lambda^a\}$ be the spectral family of $a \in (A, A^+, e)$. Let A be a real Lie algebra with Lie product $[a, b]$, $a, b \in A$, and let $\mathrm{Inn}(A, [\cdot, \cdot])$ denote the group of the inner automorphisms of $(A, [\cdot, \cdot])$ {subgroup of $\mathrm{Aut}(A, [\cdot, \cdot])$}. We say that the pair (A, A^+, e), (V, V^+, K) is an (invariant) geometric dynamical system if:

(D1) The positive cone A^+ and the order unit e in A are invariant under all inner automorphisms of $(A, [\cdot, \cdot]$—$gA^+ = A^+$, $ge = e$ for all $g \in \mathrm{Inn}(A, [\cdot, \cdot])$.

(D2) The relation $[a, b] = 0$ implies $[e_\lambda^a, b] = 0$ for all λ.

(D3) $Z_{[\cdot,\cdot]} \subseteq Z_e$, where $Z_{[\cdot,\cdot]}$ and Z_e are the centers of $(A, [\cdot, \cdot])$ and (A, A^+, e), respectively.

By definition, the (bounded) variables of the system (A, A^+, e), (V, V^+, K) are the elements of A and the states lie in K. The probability that a measurement of $a \in A$ in a state ρ yields a value in $M \subset R$ is $\langle p_M^a, \rho\rangle$, where $p_M^a = \int_M de_\lambda^a$. Accordingly, the mean value of a in ρ is the real number $\langle a, \rho\rangle$.

The requirements (D2) and (D3) are provisional and likely to be redundant. The crucial requirement is (D1) because it implies the usual Hamiltonian form of the equation of motion, existence of conservation laws, etc. Indeed, it is reasonable to assume that the time evolution is described by a one-parameter (semi)group ω_t of transformations which are symmetries of the relevant structure in A [or by the dual (semi)group ω_t' of positive transformations acting on K]. In the most regular case ω_t will preserve both structures in A (order and Lie product), i.e., $\omega_t \in \mathrm{Inn}(A, [\cdot, \cdot])$ and $\omega_t \in \mathrm{Aut}(A, A^+, e)$. Now, (D1) means nothing else but $\mathrm{Inn}(A, [\cdot, \cdot]) \subseteq \mathrm{Aut}(A, A^+, e)$ and the first condition is sufficient. As a consequence, the infinitesimal operator H of ω_t will be of the form $Ha = [a, h]$ for some $h \in A$ (the "Hamiltonian") and the equation of motion of $a(t) = \omega_t a$ becomes $da(t)/dt = [a, h]$. The second condition $[\omega_t \in \mathrm{Aut}(A, A^+, e)]$ guarantees that during the time evolution the spectrum of the variables is preserved. If we drop the condition $\omega_t \in \mathrm{Inn}(A, [\cdot, \cdot])$, we can still speak of dynamics—"non-Hamiltonian" dynamics.

Definition (D) can be further specified in order to single out the classical and quantum systems. Our experience with the associative $*$-algebras suggests that the vector lattice structure of (A, A^+, e) corresponds to the classical case, while the other extreme—antilattice structure—describes quantum systems. Recall that the vector lattice structure automatically induces in A a (unique) commutative associative multiplication with e as algebraic unit (Vulikh, 1961; Alfsen and Shultz, 1976). In the general (nonlattice) case no associative algebraic structure is implied.

There have been interesting attempts (Davies and Lewis, 1970; Edwards, 1970) to define statistical dynamical systems as a dual pair of an order-unit and a base-norm space with some mild additional requirements. It is clear, however, that such a pair cannot provide a complete description without a fully developed geometric spectral theory; what is more, we need another structure (Lie product) responsible for the usual form of dynamics. Definition (D) could be regarded as a strengthened version of the Davies and Lewis model, but it is manifestly complete and this makes it closer to the associative $*$-algebraic definitions. Unlike the latter, our definition uses two independent structures (order and Lie product), an approach leading to a surprising symmetry between classical and quantum systems (see below).

Definition (D) is discussed in more detail in Petrov (1988). Practically nothing is known on the major problem—classification of all dynamical systems in the sense of (D), for the invariantly ordered spectral Lie algebras are far from being classified and studied even in the finite-dimensional case. However, it is obvious that classical finite-dimensional geometric systems do not exist [the group $\mathrm{Aut}(A, A^+, e)$ is finite and no Lie algebra can satisfy (D1)]. This is, of course, what one should expect. On the other hand, it is a plausible conjecture that the only finite-dimensional quantum systems are represented by the Hermitian part of the full matrix algebras ordered by the cone of the positive matrices. For the first finite-dimensional results related to the interplay between spectral order and Lie algebra structure in the context of the quantum version of (D) see Hilgert (1988).

3. AN ALTERNATIVE APPROACH

I turn now to a quite different subject, taking full advantage of the two-structure character of (D) and aiming at an alternative definition of quantum dynamical systems.

Let me first remark that no explanation is known of the important fact that the quantum and classical Lie algebras are not isomorphic. It is a matter of observation that the Poisson brackets Lie algebra in classical mechanics and the commutator Lie algebra in quantum mechanics are qualitatively different. Here is the origin of all difficulties in the traditional approach to the problem of quantization. In the abstract setting of (D) one can only suspect that two cones with so different a geometry cannot live as invariant cones in the same Lie algebra.

A second observation concerns the groups $\mathrm{Aut}(A, A^+, e)$ in the classical and quantum cases. One is not able to compare directly these automorphism groups and perhaps there is no simple relation between them. The reasonable equation to ask is what is their relation to the corresponding Lie automorphism groups. In both cases (D1) holds: $\mathrm{Inn}(A, [\cdot, \cdot])_{\mathrm{cl}} \subseteq \mathrm{Aut}(A, A^+, e)_{\mathrm{cl}}$, $\mathrm{Inn}(A, [\cdot, \cdot])_{\mathrm{qu}} \subseteq \mathrm{Aut}(A, A^+, e)_{\mathrm{qu}}$. However, at least in the standard (mechanical) examples, $\mathrm{Aut}(A, A^+, e)_{\mathrm{cl}} \neq \mathrm{Inn}(A, [\cdot, \cdot]_{\mathrm{cl}}$ and the group $\mathrm{Aut}(A, A^+, e)_{\mathrm{cl}}$ is much bigger, while in the quantum case the corresponding two automorphism groups seem to coincide. So the quantum antilattice cone is "more smooth" at least with respect to its own Lie algebra.

The third and most important observation refers to the probabilistic aspect of the dynamical systems. Here the classical vector lattice structure plays an exceptional role, for only in that case can the whole space A be regarded as a genuine space of random variables in the sense of probability theory. The obvious reason is that only then is the set of the projective units (the extreme points of the order interval $\{a \in A: 0 \leqslant a \leqslant e\}$) a Boolean algebra (Vulikh, 1961; Alfsen and Shultz, 1976).

The existence of quantum systems requires abandoning this global Boolean structure, but there can be more than one way to do so. The usual way amounts to replacing the classical lattice cone by a (spectral) antilattice one while preserving its invariance with respect to the automorphism group of the corresponding (necessarily modified) Lie algebra. Is it conceivable to proceed in exactly the opposite way—to break the invariance of the positive cone and preserve its lattice geometry (as long as possible)? That is the question we now treat.

Definition (D) itself suggests what we have to do—what requirement should be accepted in place of (D1). The role of (D1) consists primarily in the implication that the group of the common order and Lie automorphisms must be as large as possible. Once this has been realized, we can replace (D1) by the opposite requirement: $\mathrm{Aut}(A, A^+, e) \subset \mathrm{Inn}(A, [\cdot, \cdot])$ (or $\mathrm{Aut}(A, A^+, e) \subset \mathrm{Aut}(A, [\cdot, \cdot])$, if necessary, the space (A, A^+, e) itself, exactly as in the classical case, being a vector lattice.

Now we have to deal with a mathematical structure radically different from the invariantly ordered Lie algebras of (D). The point is that if $\mathrm{Aut}(A, A^+, e) \subseteq \mathrm{Inn}(A, [\cdot, \cdot])$ and $\mathrm{Aut}(A, A^+, e) \neq \mathrm{Inn}(A, [\cdot, \cdot])$, we shall be forced to introduce a whole family $(A, \varphi A^+, e)$, $\varphi \in G$, of (isomorphic) vector lattices with a common order unit, parametrized by a subgroup G of $\mathrm{Inn}(A, [\cdot, \cdot])$. For example, such a family emerges inevitably in the finite-dimensional case where the group $\mathrm{Aut}(A, A^+, e)$ is finite (essentially a permutation group) and therefore no continuous subgroup of $\mathrm{Inn}(A, [\cdot, \cdot])$ can leave A^+ invariant, i.e., in general $\varphi A^+ \neq A^+$. On the other hand, no physical model can do without a continuous subgroup of $\mathrm{Inn}(A, [\cdot, \cdot])$ describing the transformation law of the dynamical variables. What is more, we must allow the cone A^+ to be acted upon by the transformations φ; otherwise we shall violate the physical equivalence between the different reference frames, whatever they might be. [In the framework of the invariant systems (D) the action or nonaction of the transformation law on A^+ makes no difference, because of the strict invariance of the cone.]

Let us remark that the family $(A, \varphi A^+, e)$ has a classical counterpart. Indeed, in the classical case again $\mathrm{Aut}(A, A^+, e) \neq \mathrm{Inn}(A, [\cdot, \cdot])$, the group $\mathrm{Aut}(A, A^+, e)$ being the bigger one. Then for $\varphi \in \mathrm{Aut}(A, A^+, e)$ we can define a new Lie product $[\cdot, \cdot]_\varphi$ by $[a, b]_\varphi = \varphi^{-1}[\varphi a, \varphi b]$ and such a construction results in a family of isomorphic Lie products {in general $[\cdot, \cdot]_\varphi \neq [\cdot, \cdot]$, and equality holds only if $\varphi \in \mathrm{Aut}(A, [\cdot, \cdot])$}. Reversing the relation between the automorphism groups, we get the family $(A, \varphi A^+, e)$ in place of $[\cdot, \cdot]_\varphi$. Following this symmetry, we say that instead of the invariant lattice cones in Lie algebras [the classical version of (D)] we now have to deal with "invariant Lie products in vector lattices." Presumably either the lattice cone or the Lie product can be invariant, but not both; two-sided invariance (that is, coincidence of the automorphism groups) requires nonlattice (if not antilattice) geometry of the cone.

4. A FAMILY OF LATTICE CONES VERSUS A SINGLE ANTILATTICE CONE

The above considerations lead to the introduction of two disjoint classes of Lie algebras equipped with a lattice order (and a fixed order unit). The first class is characterized by the relation $\mathrm{Inn}(A, [\cdot, \cdot]) \subset \mathrm{Aut}(A, A^+, e)$ and the second by the opposite relation $\mathrm{Aut}(A, A^+, e) \subset \mathrm{Inn}(A, [\cdot, \cdot])$

{or Aut$A$, A^+, e) $\subset$ Aut(A, [$\cdot$, $\cdot$]}; in both cases equality is excluded. The standard quantum systems are contained in neither of them because the quantum space (A, A^+, e) is not a vector lattice (and in addition the two automorphism groups are expected to coincide).

When our requirement is Aut(A, A^+, e) $\subset$ Aut(A, [$\cdot$, $\cdot$]) any Abelian Lie algebra can be trivially turned into a "second-class" Lie algebra. A simple construction shows that there are examples of nontrivial ("factor-like," with one-dimensional center) second-class Lie algebras (or equivalently, second-class vector lattices).

Example (E). Let A be the four-dimensional real space of the Hermitian 2×2 matrices, [$\cdot$, $\cdot$] the usual matrix commutator (multiplied by i), and I the identity matrix, and let σ_i, $i=1,2,3$, be the Pauli matrices. Let us introduce the linear bases (a_i), (b_i) given by

$$\begin{aligned} a_1 &= I+\sigma_1+\sigma_2+\sigma_3 & b_1 &= I-\sigma_1-\sigma_2-\sigma_3 \\ a_2 &= I-\sigma_1-\sigma_2+\sigma_3 & b_2 &= I+\sigma_1+\sigma_2-\sigma_3 \\ a_3 &= I-\sigma_1+\sigma_2-\sigma_3 & b_3 &= I+\sigma_1-\sigma_2+\sigma_3 \\ a_4 &= I+\sigma_1-\sigma_2-\sigma_3 & b_4 &= I-\sigma_1+\sigma_2+\sigma_3 \end{aligned}$$

Let $A_1^+, A_2^+ \subset A$ denote the lattice cones generated by (a_i) and (b_i): $A_1^+ = \{\lambda\ \mathrm{conv}(a_i)\colon \lambda \geq 0\}$, $A_2^+ = \{\lambda\ \mathrm{conv}(b_i)\colon \lambda \geq 0\}$; the convex hulls conv$(a_i)$ and conv(b_i) are bases of A_s^+. One verifies that Aut(A, A_1^+, I) = Aut(A, A_2^+, I) $\subset$ Aut(A, [$\cdot$, $\cdot$]) and consequently (A, A_1^+, I) and (A, A_2^+, I) are two second-class vector lattices on the Lie algebra (A, [$\cdot$, $\cdot$]). They are different ($A_1^+ \neq A_2^+$), but generate the same family under the action of Inn(A, [$\cdot$, $\cdot$]): {(A, φA_1^+, I): $\varphi \in$ Inn(A, [$\cdot$, $\cdot$])} = {(A, φA_2^+, I): $\varphi \in$ Inn(A, [$\cdot$, $\cdot$])}. Actually, any member of this family can be taken as an example of a second-class vector lattice. Finally, one should point out that the group Aut(A, A_1^+, I) = Aut(A, A_2^+, I) is not contained in Inn(A, [$\cdot$, $\cdot$]).

Our example thus shows that the requirement Aut(A, A^+, e) $\subset$ Inn(A, [$\cdot$, $\cdot$]) is perhaps too restrictive and in principle we have to use Aut(A, [$\cdot$, $\cdot$]) instead of Inn(A, [$\cdot$, $\cdot$]). This circumstance is quite analogous to the situation in the theory of invariant cones, where the suitable group in Inn(A, [$\cdot$, $\cdot$]) and not Aut(A, [$\cdot$, $\cdot$]) (the latter is too big and often precludes the existence of invariant cones).

According to (D), the first-class Lie algebras can be identified with classical dynamical systems. Now, what about the second-class Lie algebras? The treatment of a family of vector lattices with a common order unit as a sort of dynamical system is not as straightforward as in the invariant case (D), but nevertheless it seems to be possible. The remarkable thing is that these hypothetical dynamical systems will behave much like the familiar quantum ones possessing at the same time a considerably richer physical content. The latter is not surprising because of the existence of larger Boolean algebras in them.

In the new situation the interpretation rules adopted in (D) have to be modified in a more or less obvious manner. Fixing an element $a \in A$ is no longer sufficient to determine a unique variable; in addition to this, we have to indicate a particular cone φA^+, i.e., to fix the reference frame. Thus, the new variables are pairs $(a, \varphi A^+)$, $a \in A$, $\varphi \in G \subseteq \mathrm{Inn}(A, [\cdot, \cdot])$, which means that $a \in A$ is regarded as an element of $(A, \varphi A^+, e)$ for the chosen φ. Such variables are not uniquely determined by their mean values in all states—they depend on the additional parameters of G. Within a fixed $(A, \varphi A^+, e)$ everything remains as usual—$a \in (A, \varphi A^+, e)$ is a random variables on the Boolean algebra of the projective units associated with the cone $\varphi A^+$and the corresponding events are interpreted as outcomes of a joint measurement of some set of basic variables (with respect to the chosen reference frame).

Each vector lattice $(A, \varphi A^+, e)$ (φ fixed) is a commutative associative algebra, but the commutative multiplication is sensitive to any change ("rotation") of A^+, i.e., it is a multivalued operation in A. Operations with elements from different vector lattices $(A, \varphi A^+, e)$ do not make sense, which means in particular that the associative multiplication is either commutative or it simply does not exist—a feature characteristic for quantum systems. The multivalued commutative multiplication is defined on the whole space A and could reproduce the formal effects of the standard operator noncommutativity; in this sense we can speak of quantum behavior of the family $\{(A, \varphi A^+, e), \varphi \in G\}$. Moreover, the standard noncommutatative multiplication is physically meaningless, while the commutative multiplication inside each $(A, \varphi A^+, e)$ is induced by the corresponding Boolean structure and can be interpreted in the usual way.

The most striking properties of the family $(A, \varphi A^+, e)$ are revealed when we try to define the set of states of the new model. Following the usual prescriptions of positivity, we conclude that the states lie in the intersection of the duals of all lattice cones φA^+, $\varphi \in G$. This procedure will cut off the extreme rays of the duals of φA^+ (which again are lattice cones) and the resulting cone will possess a higher symmetry. Both effects are observed in the usual operator quantum theory, where the first is known as "uncertainty relations."

We are thus led to the hypothesis that a covariant family of vector lattices $\{(A, \varphi A^+, e), \varphi \in G\}$, where $G \subseteq \mathrm{Inn}(A, [\cdot, \cdot])$ and $(A, [\cdot, \cdot])$ is a second-class Lie algebra, does not describe dynamical systems of some unknown nature, but could possibly be used as a new description (in fact, definition) of the familiar quantum systems.

Such a hypothesis is supported by an earlier attempt to describe the simplest quantum system (spin 1/2) through a family of eight-dimensional vector lattices with a common order unit (Petrov, 1985). This first realistic example of a covariant quantum model is constructed without any reference to Lie algebra structures, but it implicitly makes use of lattice cones very similar to those from the above four-dimensional example (E). Even the oversimplified structure of (E) can be given a consistent physical meaning if we regard the triple $\sigma_i \in (A, A_s^+, I)$, $i = 1, 2, 3$, $s = 1$ or 2, as basic variables and the group $G = \text{Inn}(A, [\cdot, \cdot])$ as the corresponding transformation law. The triple of pairs $(\varphi\sigma_i, \varphi A_s^+)$ behaves like a vector, each component of which has two possible values ± 1 in every reference frame. The elements (φa_i) [resp. (φb_i)] are the four atoms of the Boolean algebra associated with $(A, \varphi A_1^+, I)$ [resp. $(A, \varphi A_2^+, I)$] interpreted as outcomes of a joint measurement of any two independent components among $(\varphi\sigma_i, \varphi A_s^+)$ [only two components are independent; for example, in (A, A_s^+, I) we have $\sigma_i\sigma_j = \pm\sigma_k$, where the multiplication is the commutative multiplication induced by the lattice cone A_s^+]. In order to have three independent components, we obviously need $2^3 = 8$ atoms and therefore a family of eight-dimensional vector lattices such as in Petrov (1985).

If we generally assume that (nontrivial) second-class Lie algebras are to be identified with quantum systems, we have to ask whether such a definition is consistent with the usual operator quantum formalism. We have to look for a factorization procedure by which the operator language [or rather its geometric version (D)] could be deduced from the covariant definition. Given the family $\{(A, \varphi A^+, e), \varphi \in G\}$, we have already observed that the set of states generates a nonlattice cone $\bigcap_G (\varphi A^+)'$ with a higher symmetry. On the other hand, we have to deal with classes $\{(a, \varphi A^+): \varphi \in G, \varphi a = a\}$ of equivalent (basic) variables and can formulate the following problem: find a simplified description in which the inner structure of these equivalence classes is ignored while the statistical characteristics of the classes as a whole are preserved (these characteristics do not depend on the choice of the representative).

In mathematical language, the problem sounds like this: find an invariant spectral cone which reproduces the spectral properties of all basic variables (in all reference frames). This is exactly what the standard antilattice cone does. So in the ideal case (in particular, when the group G is large enough), we should expect the ordered linear space (A, P, e) [P is the cone in A whose dual is $\bigcap_G (\varphi A^+)'$] to be isomorphic to the standard operator quantum description $(A_{\text{st}}, A_{\text{st}}^+, e_{\text{st}})_{\text{qu}}$. More precisely, this refers to the subspace of A generated by the basic variables, since neither the simplified model nor the standard one describes associative products between incompatible variables. For example, in Petrov (1985) the covariant description of spin 1/2 is in the space of all 2×2 matrices (regarded as eight-dimensional real space) and after the factorization we get the standard circular cone in the four-dimensional Hermitian subspace. All commutative products of the spin projections (in all reference frames) lie in the space of the anti-Hermitian matrices.

The conclusion is that the existence of a covariant description does not imply that the usual quantum definition is inadequate; it is just incomplete because in the above factorization some of the richer covariant structure is irreversibly lost. Notice, however, that this incompleteness has nothing to do with the uncertainty relations between the quantum variables, as they exist in the covariant model before its factorization. Moreover, we see that the appearance of the uncertainty relations (equivalently: nonsimplicial geometry of the set of states) is closely related to the noninvariance of the positive cone in the space of the dynamical variables.

From our new point of view the distinction between classical and quantum systems is not to be seen in the known (roughly equivalent) alternatives commutativity-noncommutativity, Booleanness-non-Booleanness, or lattice structure-antilattice structure. All these characteristics remain valid, but now they refer to the somewhat simplified standard description of the quantum systems. The truly relevant distinction is contained in the two possible inclusion relations between the automorphism groups (order and Lie automorphisms), the underlying ordered linear spaces being in both cases vector lattices. There exists a complete symmetry between classical and (covariant) quantum systems. The place of the standard invariant (nonlattice) quantum description is at the middle point where the two automorphism groups tend to coincide.

5. A COMMENT ON THE PROBLEM OF QUANTIZATION

The old problem of quantization is plagued by the fact that no isomorphism can be found between the global structures in $(A, A^+, e)_{cl}$, $(A, [\cdot, \cdot])_{cl}$ and $(A_{st}, A^+_{st}, e_{st})_{qu}$, $(A_{st}, [\cdot, \cdot]_{st})_{qu}$. If our hypothesis is correct, the whole problem is shifted and we have to deal with $(A, A^+, e)_{cl}$, $(A, [\cdot, \cdot])_{cl}$ and a quantum family $(B, \varphi B^+, e)_{qu}$, $(B, [\cdot, \cdot])_{qu}$. Now both descriptions are based on vector lattices which may well turn out to be isomorphic $[(A, A^+, e)_{cl} \sim (B, B^+, e)_{qu} \sim (B, \varphi B^+, e)_{qu}$ for any fixed $\varphi]$ and this isomorphism may provide the missing link.

The passage from the classical to the standard quantum description (when possible) will be a two-step procedure. We begin with $(A, A^+, e)_{cl}$ and $(A, [\cdot, \cdot])_{cl}$ and first look for a second-class Lie algebra $(B, [\cdot, \cdot])_{qu}$ such that $(A, A^+, e)_{cl} \sim (B, B^+, e)_{qu}$. If this problem has a solution and $(B, [\cdot, \cdot])_{qu}$ does exist, it cannot be isomorphic to $(A, [\cdot, \cdot])_{cl}$. In more concrete situations we shall have a specific group $G \subset \mathrm{Aut}(B, [\cdot, \cdot])_{qu}$ as a transformation law and therewith we come to a covariant system

$\{(B, \varphi B^+, e)_{qu}, \varphi \in G\}$. It is this family of vector lattices [each one isomorphic to $(A, A^+, e)_{cl}$ but $(B, [\cdot, \cdot])_{qu}$ not isomorphic to $(A, [\cdot, \cdot])_{cl}$] that is the quantum counterpart of the original classical system. The second step is the already mentioned factorization procedure resulting in the standard (invariant) model $(A_{st}, A_{st}^+, e_{st})_{qu}$. Clearly, the standard model cannot be obtained directly from the classical theory, because there is no longer any common (isomorphic) structure [the vector lattice structures in $(B, \varphi B^+, e)_{qu}$ are destroyed by the factorization].

Even if the intermediate structure $(B, \varphi B^+, e)_{qu}$ could not be given an independent meaning as a more complete definition of a quantum system, it still provides a new possible way of quantization via "deformation" of the classical theory.

6. CONCLUSION

The definitions formulated or just outlined lead to a long-term research program aiming at a classification of the invariant geometric dynamical systems and their covariant generalizations.

The abstract invariant definition (D) may or may not enrich our conventional understanding of dynamical systems; it is an open problem whether there are geometric models essentially different from the standard classical and quantum theories.

The hypothetical covariant quantum geometric definition (a "mirror image" of the classical one) promises much more substantial results; in a sense it makes the collection of dynamical systems twice as large as before and implies nontrivial changes in the physical status of the quantum systems.

Both the invariant lattice cones in Lie algebras and the invariant Lie products in vector lattices deserve an investigation from a purely mathematical point of view as well. They are examples of objects characterized by relations between the automorphism groups of two otherwise independent mathematical structures. Such a way of reasoning is often encountered in modern mathematics, but so far no definite results are known for the particular case when a lattice (or more generally "spectral") order is coupled with a Lie algebra structure. The different versions of this combination apparently admit immediate physical interpretation as various kinds of dynamical systems and this may give an additional stimulus for their investigation, especially in the infinite-dimensional case.

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